Unsteady Aerodynamic Forces on Slender Supersonic Aircraft with Flexible Wings and Bodies

John E. Yates* and Edmond F. E. Zeydel†
Midwest Research Institute, Kansas City, Mo.

The present paper derives generalized aerodynamic forces for slender supersonic aircraft on the basis of slender-body theory. Particular consideration is given to configurations which are spanwise flexible. To treat configurations with flexible wings and bodies, the slender wing-body problem is first reduced to a simple body problem whose solution is well known, and a "modified slender-wing problem." An integral solution of the latter is obtained, utilizing the circle theorem or method of images and a known solution of the airfoil equation for a double interval.[4] With this approach, it is not necessary to apply conformal mapping techniques, and the solution so obtained is valid for arbitrary spanwise downwash distributions. On the basis of slender-body theory, the velocity potential and, subsequently, the generalized aerodynamic forces are derived for a general class of spanwise flexible wing-body configurations.

Nomenclature

a_{nm}	= generalized coordinate, see Eq. (3.1)
A,B,C	= see Eq. (3.8)
$f(x,y)$	= planform deflection function, see Eq. (3.1)
k	= reduced frequency ($k = \omega l/U$, where ω is the flutter frequency, rad/sec)
l	= body length (ft)
$P_m(y)$	= spanwise mode shapes, see Eq. (3.2)
$P_m^{(0)}(y)$	= wing deflection modes, see Eq. (3.2)
$P(x,y)$	= differential pressure (lb/ft²)
Q_{pq}	= generalized aerodynamic force, see Eq. (4.8)
Q_{pn}^{mr}	= generalized aerodynamic force coefficient, see Eq. (4.9)
r,θ	= polar coordinates of cross-flow plane
$R(x)$	= body radius
Re	= denotes real part of a complex function
$s(x)$	= semispan of the wings
U	= forward velocity of the configuration (ft/sec)
$V(x), V(x,y)$	= downwash functions for the body and wings, respectively
$W_0(x,y), W(x,y)$	= modified downwash functions, see Eqs. (2.8) and (2.14)
w	= complex variable ($w = y + iz = re^{i\theta}$)
x,y,z	= Cartesian coordinate variables, see Figs. 1 and 2
ρ	= air density (slugs/ft³)
φ	= slender-body velocity potential
φ_1,φ_2	= components of φ, see Eq.(2.4)
$\chi(y)$	= kernel function, see Eq. (2.16)
$\chi_1(\theta)$	= see Eq. (3.12)

I. Introduction

In previous aeroelastic studies, various techniques have been used to approximate the unsteady aerodynamic forces in supersonic flow. The usual procedure is to consider only the lifting surfaces and to neglect the aerodynamic forces generated by the body. This artifice is in some cases unsatisfactory, however, in view of the trend of present day aircraft to more slender bodies and low aspect ratio wings. It is, therefore, essential to assess the importance of body flexibility on the aerodynamic forces and its influence on the flutter of slender supersonic aircraft.

It is a formidable operation indeed to consider the aeroelastic behavior of wing-body configurations on the basis of exact three-dimensional aerodynamic theory. The basic difficulty is associated with the boundary conditions as is evident from the procedure used to derive the three-dimensional velocity potential for thin wings. One usually assumes a fundamental solution of the potential equation which consists of a continuous distribution of sources or other singularities over the planform. The boundary condition that the flow shall be tangential to the surface of the wing is, in principle, easily satisfied since the source strength is proportional to the downwash. For wing-body configurations, however, it is not clear that a point relation exists between the downwash and source strength because part of the configuration surface does not coincide with the surface on which singularities are distributed. We avoid this difficulty in the present investigation by applying slender-body theory to derive a two-dimensional approximation of the velocity potential and generalized aerodynamic forces.

In previous studies, slender-body theory has been applied successfully to obtain both steady and unsteady aerodynamic forces for wing-body combinations. The well-known techniques of conformal mapping readily yield a formal solution valid for a rather arbitrary class of slender configurations.[1] To the authors' knowledge, however, practical applications of slender-body theory have been made only in the case that the local downwash is independent of the spanwise coordinate. In the present development, a method is presented for treating spanwise flexible wing-body combinations. For the circular body mid-wing configuration, the problem of treating spanwise downwash variations becomes quite tractable.

II. General Solution of Laplace's Equation for Slender Wing-Body Configurations

Consider a circular body mid-wing configuration with a typical cross section as shown in Fig. 1. We seek a solution of Laplace's equation

$$\frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \qquad (2.1)$$

subject to the boundary conditions

$$\left.\frac{\partial\varphi}{\partial r}\right|_{r=R} = V(x)\sin\theta, \text{ on the body} \qquad (2.2)$$

$$\left.\frac{\partial\varphi}{\partial z}\right|_{z=0} = V(x,y), \text{ on the wings} \qquad (2.3)$$

Presented at the Aeroelasticity Session, IAS 29th AM, New York, Jan. 23–25, 1961. Resubmitted March 12, 1962. Revised and received November 16, 1962. This work was supported by the Airframe Design Division, Bureau of Naval Weapons, USN, under Contract NO3-60-0726-c.

* Associate Analyst, Mathematics and Physics Division.
† Head, Mathematical Analysis Section.

where

$\varphi(x,y,z)$ = perturbation velocity potential‡

$V(x,y)$ = downwash on the wings

$V(x)$ = downwash along the body axis

θ = $\tan^{-1}(z/y)$

r = $\sqrt{y^2 + z^2}$

R = $R(x)$ = body radius§

Also, the potential must vanish at the leading edge of the configuration and at infinity.

To solve Eq. (2.1), let

$$\varphi(x,y,z) = \varphi_1(x,y,z) + \varphi_2(x,y,z) \tag{2.4}$$

where φ_1 and φ_2 are harmonic functions. Now, if φ_1 satisfies the boundary condition Eq. (2.2)—i.e.,

$$\left.\frac{\partial \varphi_1}{\partial r}\right|_{r=R} = V(x)\sin\theta, \text{ on the body} \tag{2.5}$$

then it readily follows from Eqs. (2.2), (2.3), and (2.4) that φ_2 must satisfy the conditions

$$\left.\frac{\partial \varphi_2}{\partial r}\right|_{r=R} = 0, \text{ on the body} \tag{2.6}$$

$$\left.\frac{\partial \varphi_2}{\partial z}\right|_{z=0} = W_0(x,y), \text{ on the wings} \tag{2.7}$$

where

$$W_0(x,y) = V(x,y) - \left.\frac{\partial \varphi_1}{\partial z}\right|_{z=0} \tag{2.8}$$

Thus φ_1 is the potential due to the body in absence of the wings, and φ_2 is the potential due to a modified downwash distribution, $W_0(x,y)$, on the wings and a rigid stationary body at the origin.

The solution for φ_1 is readily obtained by placing an isolated doublet at the origin and applying the boundary condition, Eq. (2.5). We have

$$\varphi_1(x,y,z) = -V(x)Re\left(\frac{iR^2}{w}\right) \tag{2.9}$$

where

$$w = y + iz = re^{i\varphi} \tag{2.10}$$

and Re denotes the real part of a complex function. Note that φ_1 vanishes at infinity and at the leading edge of the configuration as required. The downwash generated on the wings by the body is easily obtained from the last result. Differentiating Eq. (2.9) with respect to z and setting the latter equal to zero, we find

$$\left.\frac{\partial \varphi_1}{\partial z}\right|_{z=0} = -V(x)\frac{R^2}{y^2} \tag{2.11}$$

so that the modified downwash distribution, Eq. (2.8), becomes

$$W_0(x,y) = V(x,y) + V(x)\frac{R^2}{y^2} \tag{2.12}$$

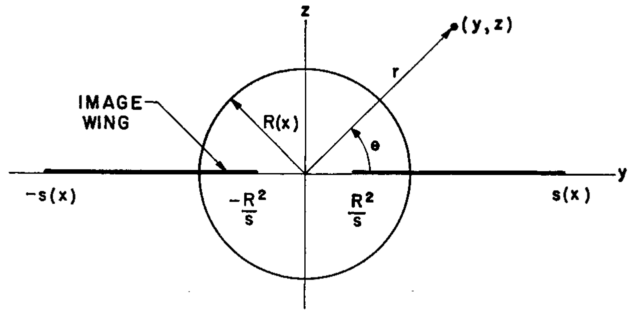

Fig. 1 Typical cross section of wing-body configuration

Now, the solution for φ_2 is easily derived by the method of images [see p. 277, Eq. (46.2) of Ref. 2] or by the circle theorem.[3] The integral equation which relates the potential on the wings and image wings (see Fig. 1) to the modified downwash distribution, Eq. (2.8), along the y axis is

$$\frac{1}{\pi}\int_{-s}^{-R^2/s} + \int_{R^2/s}^{s} \frac{\varphi_{2\eta}(x,\eta)}{\eta - y}\,d\eta = W(x,y) \tag{2.13}$$

where

$$\left.\begin{aligned} \int_{-a}^{-b} + \int_{b}^{a} f(\eta)d\eta &\equiv \int_{-a}^{-b} f(\eta)d\eta + \int_{b}^{a} f(\eta)d\eta \\ W(x,y) &= \begin{cases} W_0(x,y), & R \le |y| \le s \\ \dfrac{R^2}{y^2}W_0(x,R^2/y), & \dfrac{R^2}{s} \le |y| \le R \end{cases} \end{aligned}\right\} \tag{2.14}$$

and the subscript η denotes partial differentiation with respect to η. The result, Eq. (2.13), is the well known airfoil equation for a double interval and from Tricomi[4] we readily obtain

$$\varphi_{2_y}(x,y) = -\frac{1}{2\pi y\chi(y)}\int_{-s}^{-R^2/s} + \int_{R^2/s}^{s} \chi(\eta)W(x,\eta)\left(\frac{\eta + y}{\eta - y}\right)d\eta \tag{2.15}$$

where

$$\chi(y) = \sqrt{(s + R^2/s)^2 - (y + R^2/y)^2} \tag{2.16}$$‖

For a given downwash distribution, one can carry out the integration in Eq. (2.15) and obtain the modified slender-wing potential on the real axis. Analytic continuation of the latter yields the potential throughout the complex plane.

It is interesting to note that if the transformation $\eta = R^2/\bar{\eta}$ is introduced into Eq. (2.15) for those portions of the integrals which extend over the image wings, there results a special case of the general solution obtained by conformal mapping when the latter is transformed to physical coordinates [see Eq. (73) of Ref. 1]. The form Eq. (2.15), however, is particularly convenient from the viewpoint of numerical evaluation, since the limits of integration coincide with the branch points in the kernel function $\chi(y)$. Thus it is only necessary to choose a functional representation of the downwash Eq. (2.14) which is analytic over the wings and image wings, and tractable results are readily obtained from Eq. (2.15) by contour integration. The simplicity of this method to derive generalized unsteady aerodynamic forces for spanwise flexible wing-body configurations is clearly depicted by the example given in the following section.

‡ Since flutter is of principal interest, simple harmonic motion is assumed at the outset; hence, when we refer to the potential, downwash, or deflection, the complex amplitudes of these entities are implied.

§ Nondimensional variables are employed throughout the present report. All distances are referred to the body length l, and physical time is related to the integral U/l, where U is the forward velocity of the configuration.

‖ The kernel function $\chi(y)$ is also parametrically dependent on x through the local semispan, $s(x)$.

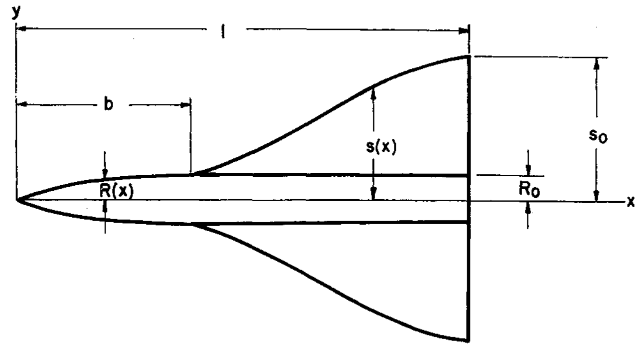

Fig. 2 Planform of wing-body configuration

III. Slender-Body Velocity Potential for Spanwise Flexible Configurations

To derive the velocity potential for spanwise flexible wing-body configurations, we consider a planform deflection function of the form

$$f(x,y) = \sum_{n=0}^{N} a_{n0}x^n + \sum_{n=0}^{N}\sum_{m=1}^{M} a_{nm}x^nP_m(y) \tag{3.1}$$

where a_{nm} is the generalized coordinate associated with the nth chordwise mode, x^n, and the mth spanwise mode

$$P_m(y) = \left\{ \begin{array}{ll} P_m^{(0)}(y) = (y^2 - R^2)^m + \dfrac{R^{2(m+1)}}{y^2}\left(\dfrac{R^2}{y^2} - 1\right)^m, & R \leq |y| \leq s \\ 0, & 0 \leq |y| \leq A \end{array} \right\} \tag{3.2}$$

Here we assume that the body cross section is approximately uniform $[\partial P_m(y)/\partial x \approx 0]$ at locations where the wings are attached. Now, the downwash is given in terms of the deflection by the relation#

$$V(x,y) = \frac{\partial f(x,y)}{\partial x} + jkf(x,y) \tag{3.3}$$

where k is the reduced frequency. From these results, together with Eqs. (2.12) and (2.14), one readily obtains a modified downwash function which is analytic over the wings and their images. We have

$$W(x,y) = V(x)(1 + R^2/y^2) + \sum_{n=0}^{N}\sum_{m=1}^{M} a_{nm}(nx^{n-1} + jkx^n)P_m^{(0)}(y), \quad \frac{R^2}{s} \leq |y| \leq s \tag{3.4}$$

where

$$V(x) = \sum_{n=0}^{N} a_{n0}(nx^{n-1} + jkx^n) \tag{3.5}$$

The first term on the right-hand side of this expression is the downwash for the spanwise rigid case; therefore, substituting Eq. (3.4) in Eq. (2.15), we get

$$\varphi_{2y}(x,y) = -V(x)\frac{\partial\chi}{\partial y} - \frac{1}{2\pi y\chi(y)}\sum_{n=0}^{N}\sum_{m=1}^{M} a_{nm}(nx^{n-1} + jkx^n) \times \left[\int_{-s}^{-R^2/s} + \int_{R^2/s}^{s}\right] \chi(\eta)P_m^{(0)}(\eta)\left(\frac{\eta + y}{\eta - y}\right)d\eta \tag{3.6}$$

In the present study, the symbols i and j both denote $\sqrt{-1}$. The distinction is made because j is associated with the complex time dependence of the problem and i is associated with the complex variable approach to the solution of Laplace's equation.

The spanwise integrals appearing in the last result are easily evaluated by contour integration. Omitting details, we write for $M = 2$

$$\varphi_{2y}(x,y) = -V(x)\frac{\partial\chi(y)}{\partial y} - \left[\frac{A(x)}{\chi(y)} + 3B(x)\chi(y) + 5C(x)\chi^3(y)\right](y - R^4/y^3) \tag{3.7}$$

where

$$\left.\begin{array}{l} A(x) = \dfrac{1}{2}\displaystyle\sum_{n=0}^{N}(nx^{n-1} + jkx^n)(s - R^2/s)^2 \times \\ \qquad \left\{-a_{n1} + a_{n2}\left[\dfrac{1}{4}(s + R^2/s)^2 - (s^2 + R^4/s^2)\right]\right\} \\ B(x) = \dfrac{1}{3}\displaystyle\sum_{n=0}^{N}(nx^{n-1} + jkx^n) \times \\ \qquad \left\{a_{n1} + \dfrac{a_{n2}}{2}[3(s^2 + R^4/s^2) - 4R^2]\right\} \\ C(x) = -\dfrac{1}{5}\displaystyle\sum_{n=0}^{N} a_{n2}(nx^{n-1} + jkx^n) \end{array}\right\} \tag{3.8}$$

But, since

$$-\frac{1}{(m+2)}\frac{\partial\chi^{m+2}(y)}{\partial y} = \chi^m(y)(y - R^4/y^3) \tag{3.9}$$

and as $\varphi_2(x,y)$ must vanish at the leading edge of the configuration, we integrate Eq. (3.7) with respect to y and obtain

$$\varphi_2(x,y) = -V(x)\chi(y) + A(x)\chi(y) + B(x)\chi^3(y) + C(x)\chi^5(y) \tag{3.10}$$

which is also the total potential, $\varphi(x,y)$, on the wings. Combining Eq. (2.9) with the analytic continuation of Eq. (3.10), one deduces the total potential on the body,

$$\begin{array}{l} \varphi(x,R,\theta) = -V(x)[\chi_1(\theta) - R\sin\theta] + A(x)[\chi_1(\theta) - 2R\sin\theta] + \\ B(x)[\chi_1^3(\theta) + 2R^3\sin 3\theta - 3R(s^2 + R^4/s^2)\sin\theta] + \\ C(x)[\chi_1^5(\theta) - 2R^5\sin 5\theta + 5R^3(s^2 + R^4/s^2)\sin 3\theta - \\ \dfrac{15R}{4}\left(s^4 + \dfrac{10R^4}{3} + \dfrac{R^8}{s^4}\right)\sin\theta] \end{array} \tag{3.11}$$

where

$$\chi_1(\theta) = \sqrt{(s + R^2/s)^2 + 4R^2\cos^2\theta} \tag{3.12}$$

IV. Derivation of the Generalized Aerodynamic Forces

Here, we derive the generalized aerodynamic forces for the planform shown in Fig. 2. It is assumed that the body and wings terminate in the base plane ($x = 1$), and that the body radius is constant ($R = R_0$) aft of the intersection ($x = b$) of the wing leading edge with the body. The maximum semispan of the wings is s_0.

Due to symmetry, the differential pressure is twice the pressure on the upper surface of the configuration. Hence, in view of the assumed geometry, we have

$$P(x,y) = 2\rho U^2\left[\frac{\partial\varphi(x,y,z)}{\partial x} + jk\varphi(x,y,z)\right] \tag{4.1}$$

where

$P(x,y)$ = differential pressure (lb/ft²)

ρ = freestream air density (slugs/ft³)

and $\varphi(x,y,z)$ is given by Eq. (3.10) on the wings and by Eq. (3.11) on the body. The generalized aerodynamic force associated with the pth chordwise mode and the qth spanwise mode is defined by

$$Q_{pq} = 2\rho U^2 \int_0^1 \int_{-s}^{s} \left[\frac{\partial\varphi(x,y,z)}{\partial x} + jk\varphi(x,y,z)\right] \times x^p P_q(y)dydx \quad (4.2)$$

which for the configuration considered can be expressed in the alternate form

$$Q_{pq} = 2\rho U^2 \int_{-s_0}^{s_0} \varphi(1,y,z)P_q(y)dy - 2\rho U^2 \times \int_0^1 (px^{p-1} - jkx^p) \int_{-s}^{s} \varphi(x,y,z)P_q(y)dydx \quad (4.3)$$

The latter is more convenient since chordwise derivatives of the potential are eliminated.

Now, for the rigid spanwise mode, $P_0(y) = 1$, we readily obtain

$$Q_{p0} = 2\rho U^2 \left[I_0(1) - \int_0^1 (px^{p-1} - jkx^p)I_0(x)dx \right] \quad (4.4)$$

where

$$\left.\begin{aligned} I_0(x) &= \int_{-s}^{s} \varphi(x,y,z)dy \\ I_0(x) &= -\frac{\pi R^2}{2} V(x) \quad \text{if } 0 \le x \le b \\ I_0(x) &= -\frac{\pi}{2} [(s - R_0^2/s)^2 + R_0^2]V(x) + \\ &\frac{\pi}{2} (s - R_0^2/s)^2 A(x) + \frac{3\pi}{8} (s - R_0^2/s)^4 B(x) + \\ &\frac{5\pi}{16} (s - R_0^2/s)^6 C(x) \quad \text{if } b \le x \le 1 \end{aligned}\right\} \quad (4.5)$$

For spanwise deflection modes we get

$$Q_{pq} = 2\rho U^2 \left[I_q(1) - \int_b^1 (px^{p-1} - jkx^p)I_q(x)dx \right], \quad q = 1,2 \quad (4.6)$$

where

$$\left.\begin{aligned} I_q(x) &= \int_{-s}^{-R_0} + \int_{R_0}^{s} \varphi(x,y)P_q(y)dy \\ &= \frac{\pi}{8} (s - R_0^2/s)^4[-V(x) + A(x)] + \\ &\frac{\pi}{16} (s - R_0^2/s)^6 B(x) + \\ &\frac{5\pi}{128} (s - R_0^2/s)^8 C(x) \quad \text{if } q = 1 \\ &= \frac{\pi}{16} (s^2 + R_0^4/s^2)(s - R_0^2/s)^4[-V(x) + \\ &A(x)] + \frac{\pi}{128} (3s^2 + 2R_0^2 + 3R^4/s^2)(s - \\ &R_0^2/s)^6 B(x) + \frac{\pi}{256} (3s^2 + 4R_0^2 + 3R_0^4/s^2) \times \\ &(s - R_0^2/s)^8 C(x) \quad \text{if } q = 2 \end{aligned}\right\} \quad (4.7)$$

Utilizing Eq. (3.8), we can express these results in terms of the generalized coordinates, a_{nm}. We have

$$Q_{pq} = \pi\rho U^2 \sum_{n=0}^{N} (a_{n0}Q_{pn}^{q0} + a_{n1}Q_{pn}^{q1} + a_{n2}Q_{pn}^{q2}), \quad p = 0,1, \ldots N, \quad q = 0,1,2 \quad (4.8)$$

where the generalized force coefficients, Q_{pn}^{qm}, are defined as follows:

$$\left.\begin{aligned} Q_{pn}^{00} &= I_{pn}^{(0)} + I_{pn}^{(2)},\ Q_{pn}^{10} = \frac{1}{4} I_{pn}^{(4)} \\ Q_{pn}^{20} &= \frac{1}{8} (I_{pn}^{(6)} + 2R_0^2 I_{pn}^{4}) \\ Q_{pn}^{11} &= \frac{1}{12} I_{pn}^{(6)},\ Q_{pn}^{21} = \frac{1}{192}(9I_{pn}^{(8)} + 16R_0^2 I_{pn}^{(6)}) \\ Q_{pn}^{22} &= \frac{1}{960} (27I_{pn}^{(10)} + 90R_0^2 I_{pn}^{(8)} + 80R_0^4 I_{pn}^{(6)}) \\ Q_{pn}^{qm} &= Q_{pn}^{mq} \end{aligned}\right\} \quad (4.9)$$

and

$$I_{pn}^{(0)} = \int_0^1 (px^{p-1} - jkx^p)(nx^{n-1} + jkx^n)R^2(x)dx - (n + jk)R_0^2 \quad (4.10)$$

$$I_{pn}^{(r)} = \int_b^1 (px^{p-1} - jkx^p)(nx^{n-1} + jkx^n)(s - R_0^2/s)^r dx - (n + jk)(s_0 - R_0^2/s_0)^r \quad r > 0 \quad (4.11)$$

Thus far the body profile ahead of the wings and the wing planform are arbitrary. Therefore, the results given here are applicable for a rather general class of slender circular body midwing configurations. The generalized aerodynamic forces are readily obtained for a particular configuration by evaluating the chordwise integrals, Eqs. (4.10) and (4.11). (See Ref. 3 where numerical results are obtained for a cone-cylinder body with delta wing.)

The results presented here are subject to the usual limitations of slender-body theory and thus are only applicable for small combinations of reduced frequency and fineness ratio. More precisely, terms of the order, $kM^2 s_0^2 \log (s_0\sqrt{M^2 - 1})$, where M is the Mach number, are neglected in the slender-body velocity potential, and at the present time studies are under way to derive an approximation which will include higher order terms.[3] The approach is based on quasi-slender-body theory,[5] and should provide a means for examining the slender-body results more critically.

References

[1] Lawrence, H. R. and Flax, A. H., "Wing-body interference at subsonic and supersonic speeds—surveys and new developments," J. Aero. Sci. 21 (1954).

[2] Durand, W. F., *Aerodynamic Theory*, Vol. II.

[3] Zeydel, E. F. E. and Yates, J. E., "Supersonic flutter of airframes with flexible bodies," Midwest Res. Inst. Rept. (April 1961).

[4] Tricomi, F. G., "The airfoil equation for a double interval," Z. für angewandte Mathematik und Physik 2 (1951).

[5] Landahl, M. T., "Forces and moments on oscillating slender wing-body combinations at sonic speed," M.I.T. Fluid Dynamics Research Group Rept. no. 56-1 (Feb. 1956).